A NOTE ON GENERATORS FOR FINITE DEPTH SUBFACTOR PLANAR ALGEBRAS

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Abstract. We show that a subfactor planar algebra of finite depth k is generated by a single $(k+1)$-box.

The main result of [KdyTpr2010] shows that a subfactor planar algebra of finite depth is singly generated with a finite presentation. If P is a subfactor planar algebra of depth k, it is shown there that a single $2k$-box generates P. It is natural to ask what the smallest t is such that a single t-box generates P. While we do not resolve this question completely, we show in this note that $t \leq k+1$ and that k does not suffice in general. All terminology and unexplained notation will be as in [KdyTpr2010].

For the rest of the paper fix a subfactor planar algebra P of finite depth k. Let $2t$ be such that it is the even number of k and $k+1$. We will show that some $2t$-box generates P as a planar algebra. The main observation is the following proposition about complex semisimple algebras. We mention as a matter of terminology that we always deal with $\mathbb{C}$-algebra anti-automorphisms and automorphisms (as opposed to those that induce a non-identity involution on the base field $\mathbb{C}$).

Proposition 1. *Let A be a complex semisimple algebra and $S : A \to A$ be an involutive algebra anti-automorphism. There exists $a \in A$ such that a and Sa generate A as an algebra.*

We pave the way for a proof of this proposition by studying two special cases. In these, n is a fixed positive integer.

Lemma 2. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. There is an algebra automorphism of $M_n(\mathbb{C})$ under which S is identified with either (i) the transpose map or (ii) the conjugate of the transpose map by the matrix*

$$J = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} (= -J^T = -J^{-1}).$$

The second case may arise only when $n = 2k$ is even (and I_k denotes, of course, the identity matrix of size k).

Proof. Let T denote the transpose map on $M_n(\mathbb{C})$. The composite map TS is then an algebra automorphism of $M_n(\mathbb{C})$ and is consequently given by conjugation with an invertible matrix, say u. Thus $Sx = (uxu^{-1})^T$. Involutivity of S implies that u is either symmetric or skew-symmetric. By Takagi's factorization (see p204 and p217 of [HrnJhn1990]), u is of the form $v^T v$ if it is symmetric and of the form $v^T J v$ if it is skew-symmetric for some invertible v. For the algebra automorphism

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of $M_n(\mathbb{C})$ given by conjugation with v, S gets identified in the symmetric case with the transpose map and in the skew-symmetric case with the conjugate of the transpose map by J. $\square$

Corollary 3. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C})$. There is a non-empty Zariski open subset of $M_n(\mathbb{C})$ such that each x in this subset is invertible and together with Sx generates $M_n(\mathbb{C})$ as an algebra.*

Proof. The elements x and Sx do not generate $M_n(\mathbb{C})$ as algebra if and only if the dimension of the span of all positive degree monomials in x and Sx is smaller than n^2. This is equivalent to saying that for any n^2 such monomials, the determinant of the $n^2 \times n^2$ matrix formed from the entries of these monomials vanishes. Since each of these entries is a polynomial in the entries of x, non-generation is a Zariski closed condition.

To show non-emptyness of the complement, it suffices, by Lemma 2, to check that when S is the transpose map or the J-conjugate of the transpose map (when n is even), some x and Sx generate $M_n(\mathbb{C})$. To see this note first that a direct computation shows that for non-zero complex numbers $\alpha_1, \cdots, \alpha_{n-1}, \beta_1, \cdots, \beta_{n-1}$, the matrix whose only non-zero entries are $\alpha_1, \cdots, \alpha_{n-1}$ on the superdiagonal and the matrix whose only non-zero entries are $\beta_1, \cdots, \beta_{n-1}$ on the subdiagonal generate $M_n(\mathbb{C})$. Taking all α_i and β_j to be 1 gives a pair of generators of $M_n(\mathbb{C})$ that are transposes of each other, while, if $n = 2k$ is even, taking all α_i to be 1 and all β_j to be 1 except for $\beta_k = -1$ gives a pair of generators of $M_n(\mathbb{C})$ such that each is the J-conjugate of the transpose of the other.

Since invertibility is also a non-empty open condition, we are done. $\square$

Lemma 4. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections under which S is identified with the map $x \oplus y \mapsto y^T \oplus x^T$.*

Proof. The map $x \oplus y \mapsto S(y^T \oplus x^T)$ is an algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ fixing the minimal central projections and is therefore given by $x \oplus y \mapsto uxu^{-1} \oplus vyv^{-1}$ for invertible u, v. Hence $S(x \oplus y) = uy^T u^{-1} \oplus vx^T v^{-1}$. By involutivity of S, we may assume that $v = u^T$. It is now easy to check that under the algebra automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by $x \oplus y \mapsto u^{-1}xu \oplus y$, S is identified with $x \oplus y \mapsto y^T \oplus x^T$. $\square$

In proving the analogue of Corollary 3 for $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, we will need the following lemma.

Lemma 5. *Let A and B be finite dimensional complex unital algebras and let $a \in A$ and $b \in B$ be invertible. Then, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $a \oplus \lambda b \in A \oplus B$ contains both a and b.*

Proof. We may assume that $\lambda \neq 0$ and then it suffices to see that a is expressible as a polynomial in $a \oplus \lambda b$. Note that since $a \oplus \lambda b$ is invertible and $A \oplus B$ is finite dimensional, the algebra generated by $a \oplus \lambda b$ is actually unital. In particular, it makes sense to evaluate any complex univariate polynomial on $a \oplus \lambda b$.

Let $p(X)$ and $q(X)$ be the minimal polynomials of a and b respectively. By invertibility of a and b, neither p nor q has 0 as a root. The minimal polynomial of λb is $q(\frac{X}{\lambda})$. Unless λ is the quotient of a root of p and a root of q, $p(X)$ and

$q(\frac{X}{\lambda})$ will have no common roots and therefore be coprime. So there will exist a polynomial $r(X)$ that is divisible by $q(\frac{X}{\lambda})$ but is 1 modulo $p(X)$. Thus $r(a \oplus \lambda b) = a$, as desired. □

Corollary 6. *Let S be an involutive algebra anti-automorphism of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ that interchanges the two minimal central projections. There is a non-empty Zariski open subset of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ such that each $x \oplus y$ in this subset is invertible and together with $S(x \oplus y)$ generates $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra.*

Proof. As in the proof of Corollary 3, the set of $x \oplus y \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ such that $x \oplus y$ and $S(x \oplus y)$ do not generate $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra is Zariski closed. Thus the set of invertible elements in its complement is Zariski open.

To show non-emptyness, it suffices, by Lemma 4, to check that some invertible $x \oplus y$ and $y^T \oplus x^T$ generate $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ as an algebra. Note that by Corollary 3, there is an invertible $x \in M_n(\mathbb{C})$ such that x and x^T generate $M_n(\mathbb{C})$. By Lemma 5, for all but finitely many $\lambda \in \mathbb{C}$, the algebra generated by $x \oplus \lambda x$ contains $x \oplus 0$ and $0 \oplus x$ and similarly the algebra generated by $\lambda x^T \oplus x^T$ contains $x^T \oplus 0$ and $0 \oplus x^T$. Thus the algebra generated by $x \oplus \lambda x$ and $\lambda x^T \oplus x^T$ is the whole of $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. □

Proof of Proposition 1. Let $\hat{A}$ denote the (finite) set of all inequivalent irreducible representations of A and for $\pi \in \hat{A}$, let d_π denote its dimension. Since S is an involutive anti-automorphism, it acts as an involution on the set of minimal central projections of A. It is then easy to see that there exist subsets $\hat{A}_1$ and $\hat{A}_2$ of $\hat{A}$ and an identification

$$A \to \bigoplus_{\pi \in \hat{A}_1} M_{d_\pi}(\mathbb{C}) \oplus \bigoplus_{\pi \in \hat{A}_2} (M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C}))$$

such that each summand is S-stable.

Now, by Corollaries 3 and 6, in each summand of the above decomposition, either $M_{d_\pi}(\mathbb{C})$ or $M_{d_\pi}(\mathbb{C}) \oplus M_{d_\pi}(\mathbb{C})$, there is an invertible element which together with its image under S generates that summand.

Finally, an inductive application of Lemma 5 shows that if a is a general linear combination of these generators, then a and Sa generate A as an algebra. □

Our main result now follows easily.

Proposition 7. *Let P be a subfactor planar algebra of finite depth k. Let $2t$ be the even number in $\{k, k+1\}$. Then P is generated by a single $2t$-box.*

Proof. It clearly suffices to see that there is a $2t$-box such that the planar subalgebra generated by it contains P_{2t}, for then, this generated planar subalgebra contains P_k as well (taking the right conditional expectation if $2t = k+1$) and hence is the whole of P.

Let S denote the map $Z_{(R_{2t})^{\circ t}} : P_{2t} \to P_{2t}$ which is an involutive anti-automorphism of the semisimple algebra P_{2t}. By Proposition 1, there is an $a \in P_{2t}$ such that a and Sa generate P_{2t} as an algebra. Since the planar subalgebra generated by a certainly contains Sa, it follows that it contains all of P_{2t}. □

We finish by showing that $k+1$ might actually be needed.

Example 8. *Let $P = P(V)$ be the tensor planar algebra (see* [Jns1999]*) for details) of a vector space V of dimension greater than 1. It is easy to see that $depth(P) = 1$. However, given any $a \in P_1 = End(V)$, if Q is the planar subalgebra of P generated by a, a little thought shows that Q_1 is just the algebra generated by a and is hence abelian while P_1 is not.*

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